Two Strange Constructions in the Euclidean Plane

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Abstract

We present two new constructions in the usual euclidean plane. We only deal with 'Grecian Geometry', with this phrase we mean elementary geometry in the two-dimensional space $\mathbb{R}^2$. We describe and prove two propositions about 'projections'. The proofs need only elementary analytical knowledge.

The reader may find the foundations and assumptions of the following propositiones in many books about plane geometry, for instance in [1], p.1-29 . Or you can look in [2] , [3] , [4] , [5] , [6], [7] . See also [8], p.224-234 .

Proposition 1. Let us take $\mathbb{R}^2 = \{(x|y)|\ x, y \in \mathbb{R}\}$, the two-dimensional euclidean plane, with the horizontal x-axis and the vertical y-axis. Assume two parallel straight lines G_S and G_T. Assume a third line L , not parallel to G_S, G_T, respectively, with the property that L does not meet the origin $(0|0)$. The intersection of L with G_S is called $S = (x_S|y_S)$, and the intersection of L with G_T is called $T = (x_T|y_T)$. Note that, in the case that G_S, G_T are distinct, the three points $(0|0), S, T$ are not collinear. We can draw two lines Z_S and Z_T, Z_S connects the origin $(0|0)$ and S , and Z_T connects $(0|0)$ and T . Z_S and Z_T are distinct if G_S and G_T are distinct. Now we distinguish two cases (A) and (B) , but note that they overlap.

(A): In the case that G_S and G_T are not parallel to the horizontal x-axis, we have two intersections a_S, a_T of G_S and G_T, respectively, with the x-axis. Then there is an unique point $P_{hor} = (x_{hor}|y_{hor})$ on L , such that $(x_{hor} - a_S|y_{hor}) \in Z_T$, and $(x_{hor} - a_T|y_{hor}) \in Z_S$.

(B): In the case that G_S and G_T are not parallel to the vertical y-axis, we have two intersections b_S, b_T of G_S and G_T, respectively, with the y-axis. Then there is an unique point $P_{ver} = (x_{ver}|y_{ver})$ on L , such that $(x_{ver}|y_{ver} - b_S) \in Z_T$, and $(x_{ver}|y_{ver} - b_T) \in Z_S$.

Before reading the proof of the proposition you should take a look on Picture 1 .
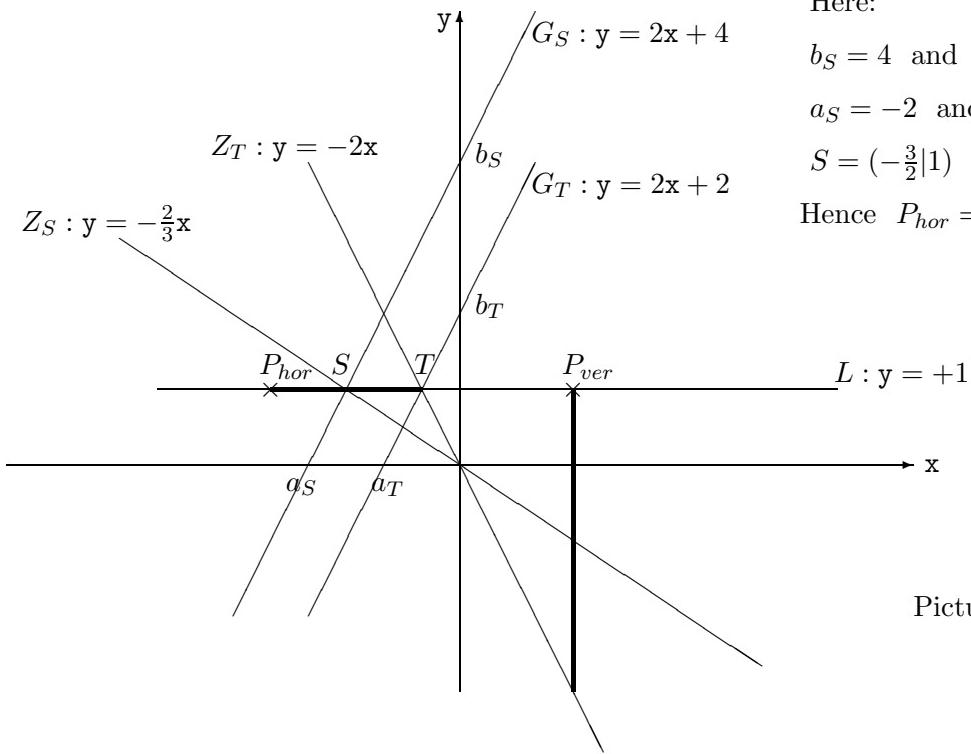
Here:

$b_S = 4$ and $b_T = 2$,

$a_S = -2$ and $a_T = -1$,

$S = (-\frac{3}{2}|1)$ and $T = (-\frac{1}{2}|1)$.

Hence $P_{hor} = (-\frac{5}{2}|1)$ and $P_{ver} = (\frac{3}{2}|1)$.

Picture 1

Proof. Note that, if $G_S = G_T$, the proposition is trivial. Hence we assume that G_S, G_T are distinct. We describe the parallel straight lines G_S, G_T with equations

$$G_S := \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid e \cdot \mathbf{y} = m \cdot \mathbf{x} + b_S\} \quad \text{and} \quad G_T := \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid e \cdot \mathbf{y} = m \cdot \mathbf{x} + b_T\} ,$$

with $e, m, b_S, b_T \in \mathbb{R}$, $(m, e) \neq (0, 0)$. Without less of generality let either be ($e = 0$ and $m = 1$) or ($e = 1$).

The straight line L can be described with two numbers $w_1, w_2 \in \mathbb{R}$, $(w_1, w_2) \neq (0, 0)$.

$$L := \{ (\mathbf{x}_S|\mathbf{y}_S) + t \cdot (w_1|w_2) \mid t \in \mathbb{R} \} = \{ (\mathbf{x}_T|\mathbf{y}_T) + t \cdot (w_1|w_2) \mid t \in \mathbb{R} \} ,$$

with $w_1 \cdot \mathbf{y}_T \neq w_2 \cdot \mathbf{x}_T$, and $w_1 \cdot \mathbf{y}_S \neq w_2 \cdot \mathbf{x}_S$, (because $(0|0) \notin L$) , and with $e \cdot w_2 \neq m \cdot w_1$, (because L is not parallel to G_S and G_T , respectively) .

Lemma 1. In the case of (A), (that means that G_S and G_T are not parallel to the horizontal x-axis), we have $m \neq 0$, and $a_S = -b_S/m$, $a_T = -b_T/m$. Then there are uniquely three numbers $\varrho, \alpha, \beta \in \mathbb{R}$ which solve the system of four linear equations

$$(1) \quad \mathbf{x}_S + \varrho \cdot w_1 - a_S = \alpha \cdot \mathbf{x}_T \quad , \qquad (2) \quad \mathbf{y}_S + \varrho \cdot w_2 = \alpha \cdot \mathbf{y}_T \quad ,$$

$$(3) \quad \mathbf{x}_S + \varrho \cdot w_1 - a_T = \beta \cdot \mathbf{x}_S \quad , \qquad (4) \quad \mathbf{y}_S + \varrho \cdot w_2 = \beta \cdot \mathbf{y}_S \quad .$$

In the case of (B), (that means that G_S and G_T are not parallel to the vertical y-axis), there are uniquely three numbers $\widetilde{\varrho}, \widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}$ which solve the system of four equations

$$\widetilde{(1)} \quad \mathbf{y}_S + \widetilde{\varrho} \cdot w_2 - b_S = \widetilde{\alpha} \cdot \mathbf{y}_T \quad , \qquad \widetilde{(2)} \quad \mathbf{x}_S + \widetilde{\varrho} \cdot w_1 = \widetilde{\alpha} \cdot \mathbf{x}_T \quad ,$$

$$\widetilde{(3)} \quad \mathbf{y}_S + \widetilde{\varrho} \cdot w_2 - b_T = \widetilde{\beta} \cdot \mathbf{y}_S \quad , \qquad \widetilde{(4)} \quad \mathbf{x}_S + \widetilde{\varrho} \cdot w_1 = \widetilde{\beta} \cdot \mathbf{x}_S \quad .$$

Proof. In case (A) the two equations (1),(2) yield $\varrho_{[1]}$, and the two equations (3),(4) yield $\varrho_{[2]}$,

$$\varrho_{[1]} = \frac{\mathbf{y}_S \cdot \mathbf{x}_T - \mathbf{x}_S \cdot \mathbf{y}_T + a_s \cdot \mathbf{y}_T}{w_1 \cdot \mathbf{y}_T - w_2 \cdot \mathbf{x}_T} \quad \text{and} \quad \varrho_{[2]} = \frac{\mathbf{y}_S \cdot a_T}{w_1 \cdot \mathbf{y}_S - w_2 \cdot \mathbf{x}_S} \quad .$$

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Because of $(\mathbf{x}_S|\mathbf{y}_S), (\mathbf{x}_T|\mathbf{y}_T) \in L$, there is a $\breve{t} \in \mathbb{R}$ such that $(\mathbf{x}_T|\mathbf{y}_T) = (\mathbf{x}_S|\mathbf{y}_S) + \breve{t} \cdot (w_1, w_2)$, hence $w_1 \cdot \mathbf{y}_T - w_2 \cdot \mathbf{x}_T = w_1 \cdot \mathbf{y}_S - w_2 \cdot \mathbf{x}_S$. And with $(\mathbf{x}_S|\mathbf{y}_S) \in G_S$, $(\mathbf{x}_T|\mathbf{y}_T) \in G_T$ follows easily that $\mathbf{y}_S \cdot \mathbf{x}_T - \mathbf{x}_S \cdot \mathbf{y}_T + a_s \cdot \mathbf{y}_T = \mathbf{y}_S \cdot a_T$, hence $\varrho_{[1]} = \varrho_{[2]} =: \varrho$.

In the case of (B) the two equations $\widetilde{(1)}\widetilde{(2)}$ yield $\widetilde{\varrho}_{[1]}$, and $\widetilde{(3)}\widetilde{(4)}$ yield $\widetilde{\varrho}_{[2]}$,

$$\widetilde{\varrho}_{[1]} = \frac{\mathbf{y}_T \cdot \mathbf{x}_S - \mathbf{x}_T \cdot \mathbf{y}_S + b_s \cdot \mathbf{x}_T}{w_2 \cdot \mathbf{x}_T - w_1 \cdot \mathbf{y}_T} \quad \text{and} \quad \widetilde{\varrho}_{[2]} = \frac{\mathbf{x}_S \cdot b_T}{w_2 \cdot \mathbf{x}_S - w_1 \cdot \mathbf{y}_S}.$$

and with similar steps as only just follows $\widetilde{\varrho}_{[1]} = \widetilde{\varrho}_{[2]} =: \widetilde{\varrho}$, and the lemma is proved. $\quad\square$

To finish the proof of proposition 1 we set in the
case (A): $P_{hor} = (\mathbf{x}_{hor}|\mathbf{y}_{hor}) := (\mathbf{x}_S|\mathbf{y}_S) + \varrho \cdot (w_1|w_2)$, and in the
case (B): $P_{ver} = (\mathbf{x}_{ver}|\mathbf{y}_{ver}) := (\mathbf{x}_S|\mathbf{y}_S) + \widetilde{\varrho} \cdot (w_1|w_2)$.
The uniqueness of P_{hor} and P_{ver} is trivial, for instance, for a non vertical L, the horizontal distance (with signs) from a point on L to Z_S or Z_T, respectively,

$$\mathbf{x} \longmapsto \text{ the horizontal distance (with sign) from a point } (\mathbf{x}|\mathbf{y}) \text{ on } L \text{ to } Z_S \quad \text{and}$$

$$\mathbf{x} \longmapsto \text{ the horizontal distance (with sign) from a point } (\mathbf{x}|\mathbf{y}) \text{ on } L \text{ to } Z_T, \quad \text{respectively,}$$

are strictly monotone functions. This was the last what we had to do to prove the proposition. $\quad\square$

For completeness, we write down other representations of P_{hor} and P_{ver}, respectively.
Note that if G_S, G_T are not parallel to the vertical y-axis, they have equations

$$G_S = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{y} = m \cdot \mathbf{x} + b_S\} \quad \text{and} \quad G_T = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{y} = m \cdot \mathbf{x} + b_T\},$$

and if G_S, G_T are vertical, they have equations

$$G_S = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{x} = -b_S =: a_S\} \quad \text{and} \quad G_T = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{x} = -b_T =: a_T\}.$$

If L is not vertical, we have $w_1 \neq 0$, and

$$L = \{(\mathbf{x}_S|\mathbf{y}_S) + t \cdot (w_1|w_2) \mid t \in \mathbb{R}\} = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{y} = m_L \cdot \mathbf{x} + b_L\},$$

with $m_L := w_2/w_1$ and $b_L := \mathbf{y}_S - \mathbf{x}_S \cdot m_L$, $(b_L \neq 0$, because $(0|0) \notin L)$.

If L is vertical, we set $a_L := \mathbf{x}_S = \mathbf{x}_T$, and $L = \{(\mathbf{x}|\mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{x} = a_L\}$.

Now assume that neither L nor G_S, G_T are parallel to one of the axes. Then

$$P_{hor} = (\mathbf{x}_{hor}|\mathbf{y}_{hor}) = \left(\frac{\mathbf{x}_T \cdot b_L + \mathbf{y}_T \cdot a_S}{b_L} \;\middle|\; m_L \cdot \frac{\mathbf{x}_T \cdot b_L + \mathbf{y}_T \cdot a_S}{b_L} + b_L\right)$$

$$= \left(\frac{\mathbf{x}_S \cdot b_L + \mathbf{y}_S \cdot a_T}{b_L} \;\middle|\; m_L \cdot \frac{\mathbf{x}_S \cdot b_L + \mathbf{y}_S \cdot a_T}{b_L} + b_L\right) =$$

$$\left(\frac{b_L^2 \cdot m + b_S \cdot b_T \cdot m_L - m \cdot b_L \cdot (b_S + b_T)}{b_L \cdot m \cdot (m - m_L)} \;\middle|\; \frac{b_L^2 \cdot m^2 + b_S \cdot b_T \cdot m_L^2 - m \cdot m_L \cdot b_L \cdot (b_S + b_T)}{b_L \cdot m \cdot (m - m_L)}\right),$$

$$P_{ver} = (\mathbf{x}_{ver}|\mathbf{y}_{ver}) = \left(\frac{\mathbf{x}_T \cdot (b_L - b_S)}{b_L} \;\middle|\; m_L \cdot \frac{\mathbf{x}_T \cdot (b_L - b_S)}{b_L} + b_L\right)$$

$$= \left(\frac{\mathbf{x}_S \cdot (b_L - b_T)}{b_L} \;\middle|\; m_L \cdot \frac{\mathbf{x}_S \cdot (b_L - b_T)}{b_L} + b_L\right)$$

$$= \left(\frac{(b_L - b_T) \cdot (b_L - b_S)}{b_L \cdot (m - m_L)} \;\middle|\; \frac{m_L \cdot (b_S \cdot b_T - b_L \cdot b_S - b_L \cdot b_T) + b_L^2 \cdot m}{b_L \cdot (m - m_L)}\right).$$

Now assume that G_S, G_T are not parallel to one of the axes, and L is horizontal. (See the previous picture, too.) Then we have an equation $L : y = b_L$, and we get

$$P_{hor} = (x_{hor}|y_{hor}) = (x_T + a_S \mid b_L) = (x_S + a_T \mid b_L) = \left(\frac{b_L - b_S - b_T}{m} \mid b_L \right) ,$$

$$P_{ver} = \left(\frac{x_T \cdot (b_L - b_S)}{b_L} \mid b_L \right) = \left(\frac{x_S \cdot (b_L - b_T)}{b_L} \mid b_L \right) = \left(\frac{(b_L - b_S) \cdot (b_L - b_T)}{b_L \cdot m} \mid b_L \right) .$$

Assume that G_S, G_T are not parallel to one of the axes, and L is vertical. Then we have an equation $L : x = a_L$, and we get

$$P_{hor} = (x_{hor}|y_{hor}) = \left(a_L \mid m \cdot a_L + b_S + b_T + \frac{b_S \cdot b_T}{a_L \cdot m} \right) = \left(a_L \mid (m + \frac{b_S}{a_L}) \cdot (a_L + \frac{b_T}{m}) \right) ,$$

$$P_{ver} = (x_{ver}|y_{ver}) = (a_L \mid m \cdot a_L + b_T + b_S) .$$

Now assume that G_S, G_T are parallel to the horizontal x-axis, and L is not parallel to the y-axis (and, of course, not parallel to the x-axis, too) . Then we get no P_{hor} , and

$$
\begin{aligned}
P_{ver} = (x_{ver}|y_{ver}) &= \left(\frac{x_T \cdot (b_L - b_S)}{b_L} \mid m_L \cdot \frac{x_T \cdot (b_L - b_S)}{b_L} + b_L \right) \\
&= \left(\frac{x_S \cdot (b_L - b_T)}{b_L} \mid m_L \cdot \frac{x_S \cdot (b_L - b_T)}{b_L} + b_L \right) \\
&= \left(\frac{(b_T - b_L) \cdot (b_L - b_S)}{b_L \cdot m_L} \mid \frac{b_L \cdot b_S + b_L \cdot b_T - b_S \cdot b_T}{b_L} \right) .
\end{aligned}
$$

If we assume that G_S, G_T are parallel to the horizontal x-axis, and L is parallel to the y-axis, then we get no P_{hor} , of course, and

$$P_{ver} = (x_{ver}|y_{ver}) = (a_L \mid b_S + b_T) .$$

Now assume that G_S, G_T are parallel to the vertical y-axis, and L is not parallel to the x-axis (and, of course, not parallel to the y-axis, too) . Then we get no P_{ver} , and

$$
\begin{aligned}
P_{hor} = (x_{hor}|y_{hor}) &= \left(\frac{a_T \cdot b_L + y_T \cdot a_S}{b_L} \mid m_L \cdot \frac{a_T \cdot b_L + y_T \cdot a_S}{b_L} + b_L \right) \\
&= \left(\frac{a_S \cdot b_L + y_S \cdot a_T}{b_L} \mid m_L \cdot \frac{a_S \cdot b_L + y_S \cdot a_T}{b_L} + b_L \right) \\
&= \left(\frac{b_L \cdot (a_S + a_T) + m_L \cdot a_S \cdot a_T}{b_L} \mid m_L \cdot x_{hor} + b_L \right) .
\end{aligned}
$$

And finally if we assume that G_S, G_T are parallel to the y-axis, and L is parallel to the x-axis, we get

$$P_{hor} = (x_{hor}|y_{hor}) = (a_S + a_T \mid b_L) .$$

Remark 1. Note a few special trivial cases.
Assume that G_S, G_T are not parallel to the horizontal x-axis (case (A)) .
If $S = (a_S|0)$, then we have $Z_S = \text{x-axis}$ and $P_{hor} = S = (a_S|0)$.
If $T = (a_T|0)$, then we have $Z_T = \text{x-axis}$ and $P_{hor} = T = (a_T|0)$.
Assume now that G_S, G_T are not parallel to the vertical y-axis (case (B)) .
If $S = (0|b_S)$, then we have $Z_S = \text{y-axis}$ and $P_{ver} = S = (0|b_S)$.
If $T = (0|b_T)$, then we have $Z_T = \text{y-axis}$ and $P_{ver} = T = (0|b_T)$.

Now we describe another proposition which seems to be more general, but indeed it is equivalent, see lemma 2. Because we proved proposition 1 , proposition 2 also is true.

Proposition 2. Let us take $\mathbb{R}^2$, the two-dimensional euclidean plane. Assume two parallel straight lines G_S and G_T. Assume a third line L , not parallel to G_S and G_T , respectively. The intersection of L with G_S is called S, and the intersection of L with G_T is called T . Assume a fourth line $Axis$, $Axis \neq L$, and $Axis$ is not parallel to G_S and G_T . The intersection of $Axis$ with G_S is called S_{Axis}, and the intersection of $Axis$ with G_T is called T_{Axis} . Further we choose a point $Origin$ on $Axis \backslash L$. We can draw two straight lines Z_S and Z_T , Z_S connects $Origin$ and S , and Z_T connects $Origin$ and T . As every line, Z_S and Z_T , respectively, divide the plane in two halfplanes. Z_S and Z_T are distinct if G_S and G_T are distinct.
Then there is an unique point $\mathsf{P} \in L$ with the following properties:
We draw the straight line $Axis_\mathsf{P}$ which meets P and which is parallel to $Axis$.
We have to distinguish three cases, the main one (1) and two trivial ones (2),(3) .
(1) $S_{Axis} \neq S$ and $T_{Axis} \neq T$.
Then $Axis_\mathsf{P} \neq Axis$, the intersection of $Axis_\mathsf{P}$ with Z_S is called S_P,   and the intersection of $Axis_\mathsf{P}$ with Z_T is called T_P .
Then the distance of S_{Axis} and $Origin$ is equal to the distance of P and T_P, and the distance of T_{Axis} and $Origin$ is equal to the distance of P and S_P . Furthermore, S_{Axis} and P are on the same side of Z_T , and T_{Axis} and P are on the same side of Z_S , respectively. (See Picture 2) .
(2) $S_{Axis} = S$. (Hence, if $G_S \neq G_T$ then $T_{Axis} \neq T$) .
Then $\mathsf{P} := S_{Axis} = S$,   and $Axis_\mathsf{P} = Axis = Z_S$.
The intersection of $Axis_\mathsf{P}$ with Z_T is $T_\mathsf{P} := Origin$.
Then, by triviality, the distance of S_{Axis} and $Origin$ is equal to the distance of P and T_P, and furthermore, by triviality, S_{Axis} and P are on the same side of Z_T .
(3) $T_{Axis} = T$. (Hence, if $G_S \neq G_T$ then $S_{Axis} \neq S$) .
Then $\mathsf{P} := T_{Axis} = T$,   and $Axis_\mathsf{P} = Axis = Z_T$.
The intersection of $Axis_\mathsf{P}$ with Z_S is $S_\mathsf{P} := Origin$.
Then, by triviality, the distance of T_{Axis} and $Origin$ is equal to the distance of P and S_P, and furthermore, by triviality, T_{Axis} and P are on the same side of Z_S .

Picture 2
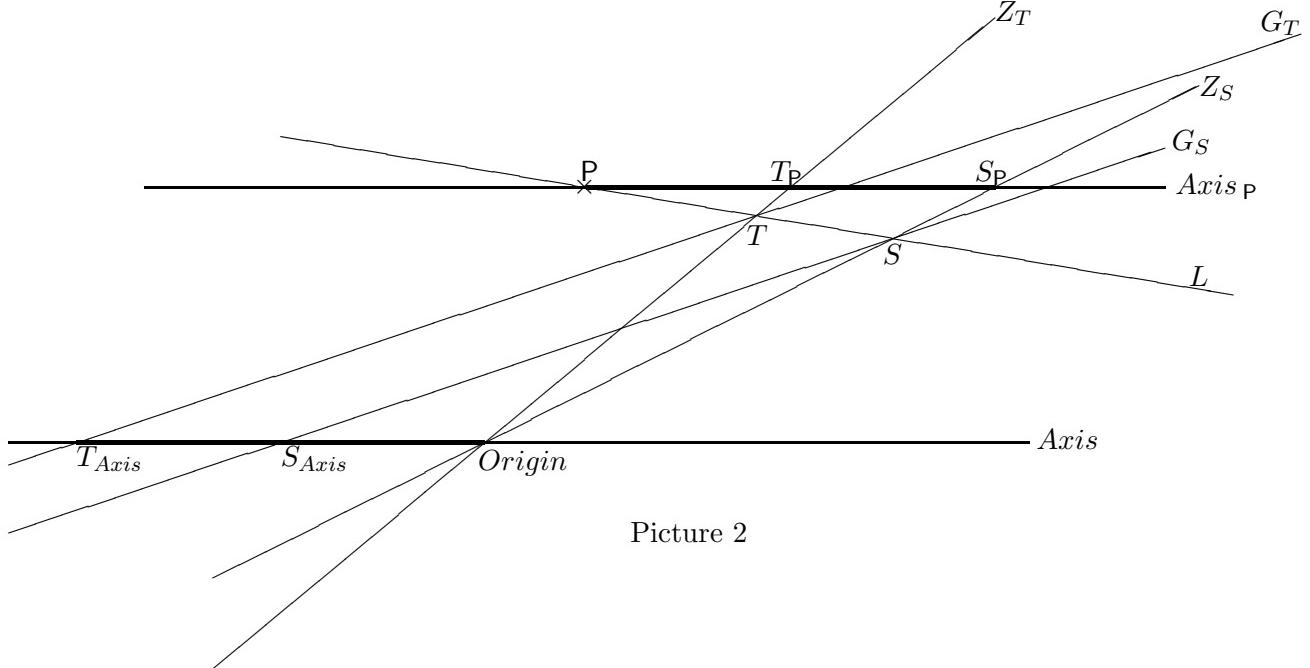
Lemma 2.   We have that   proposition 1 $\iff$ proposition 2 .

Proof.   proposition 1 $\impliedby$ proposition 2:
Obviously, the two situations which are described in proposition 1 are special cases of the general situation in proposition 2 . More detailed, we have $Origin := (0|0)$ and if we define $Axis := \text{x-axis}$ we get $P_{hor} = \mathsf{P}$, and $Axis := \text{y-axis}$ yields $P_{ver} = \mathsf{P}$, respectively.
proposition 1 $\implies$ proposition 2:
With an easy transformation of coordinates, we get $(0|0) = Origin$, and $\text{x-axis} = Axis$, hence $\mathsf{P} = P_{hor}$. $\square$

Now follows another piece of 'Grecian Geometry'.

Proposition 3. Let us again take $\mathbb{R}^2 = \{(\text{x}|\text{y})|\ \text{x}, \text{y} \in \mathbb{R}\}$. with the horizontal x-axis and the vertical y-axis. Consider the two parallel lines G, P (P means 'projection line') , with the property that G does not meet $(0|0)$. Assume a fixed $\varepsilon \in \mathbb{R}$. Let us choose a point $(\widehat{\text{x}}|\widehat{\text{y}})$ on G , $\widehat{\text{y}} \neq 0$, such that neither the line that connects $(0|0)$ and $S := (\widehat{\text{x}} - \varepsilon|\widehat{\text{y}})$, nor the line that connects $(0|0)$ and $T := (\widehat{\text{x}} + \varepsilon|\widehat{\text{y}})$ is parallel to G and P . We call $\overline{S}$ the projection of S on the line P , and $\overline{T}$ the projection of T on the line P. (That means that the three points $(0|0), S, \overline{S}$, and the three points $(0|0), T, \overline{T}$, respectively, are collinear, $\overline{S}, \overline{T} \in P$.) The four points $\overline{S}, \overline{T}, -\overline{S}, -\overline{T}$ are the corners of a parallelogram. We call $'\nu'$ the intersection of the line that connects $\overline{T}$ and $-\overline{S}$ with the horizontal x-axis. For the claim we distinguish two disjoint cases:
(A)   If P and G are parallel to the vertical y-axis, then ν depends only on ε and on the intersections of the horizontal x-axis with G and P, respectively.
(B)   If P and G are not parallel to the vertical y-axis, then ν depends only on ε and on the intersections of the vertical y-axis with G and P, respectively.   (See Picture 3) .

In other words we claim that the value of ν does not depend on the choice of $(\widehat{\text{x}}|\widehat{\text{y}})$ on G , and also, (in case (B)) , that ν does not depend on the slope of G and P .

We fix two lines P, G (with slope 2) ,
and $\varepsilon := 4$. We choose $(\widehat{\text{x}}|\widehat{\text{y}}) := (0|4)$.
We get $\overline{S} = (-\frac{2}{3}|\frac{2}{3})$, $\overline{T} = (-2|-2)$.
Hence we get $\nu = 2$.

$G : \text{y} = 2 \cdot \text{x} + 4$

$P : \text{y} = 2 \cdot \text{x} + 2$

$\text{y} = \frac{1}{2} \cdot \text{x} - 1$

Picture 3

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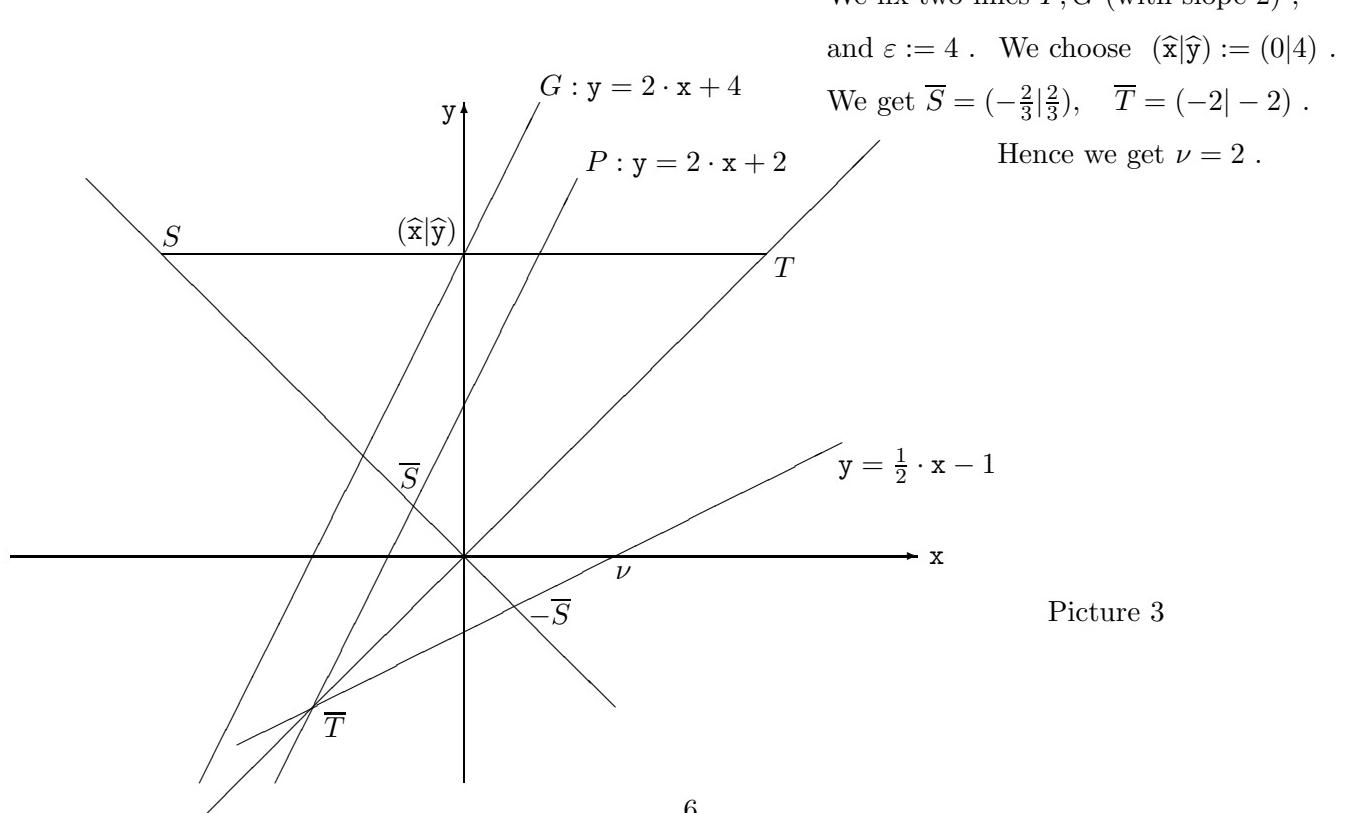
Proof. First the trivial cases. If P meets the origin $(0|0)$, then $(0|0) = \overline{T} = \overline{S}$, and the parallelogram collapses into a single point $(0|0) = \nu$. If $\varepsilon = 0$, then $\overline{T} = \overline{S}$, and the parallelogram degenerates to a line between $\overline{T}$ and $-\overline{S}$, that meets $(0|0) = \nu$.

Hence we assume that P does not meet the origin $(0|0)$, and we take (without loss of generality) an $\varepsilon > 0$. We distinguish between vertical G, P and not vertical G, P . Thus assume vertical lines G and P with equations $G: \mathbf{x} = r$ and $P: \mathbf{x} = p$. After choosing a point $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ on G , $\widehat{\mathbf{y}} \neq 0$, we can compute $\overline{S}$ and $\overline{T}$. Because G does not meet $(0|0)$ we have $r \neq 0$, and some easy calculations yield $\nu = p \cdot \varepsilon / r$.

In the case that G, P are not vertical they have a slope $m \in \mathbb{R}$, and there are equations

$$G: \mathbf{y} = m \cdot \mathbf{x} + b_G \qquad \text{and} \qquad P: \mathbf{y} = m \cdot \mathbf{x} + b_P$$

with $m, b_G, b_P \in \mathbb{R}$, $b_G, b_P \neq 0$. After choosing a point $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ on G , $\widehat{\mathbf{y}} \neq 0$, we get with elementary calculations

$$\overline{S} = \frac{b_P}{b_G + m \cdot \varepsilon} \cdot (\widehat{\mathbf{x}} - \varepsilon \mid \widehat{\mathbf{y}}) \qquad \text{and} \qquad \overline{T} = \frac{b_P}{b_G - m \cdot \varepsilon} \cdot (\widehat{\mathbf{x}} + \varepsilon \mid \widehat{\mathbf{y}}) \; .$$

Some more calculations yield the formula

$$\mathbf{y} = \frac{m \cdot \widehat{\mathbf{x}} + b_G}{\widehat{\mathbf{x}} \cdot b_G + m \cdot \varepsilon^2} \; \cdot \; [\, b_G \cdot \mathbf{x} - b_P \cdot \varepsilon \,]$$

for a non vertical straight line that intersects $\overline{T}$ and $-\overline{S}$, and finally we get $\nu = b_P \cdot \varepsilon / b_G$. If the line that connects $\overline{T}$ and $-\overline{S}$ is vertical we get the same formula for ν , and the proof of the proposition is complete. $\square$

Remark 2. The four points $\overline{S}, \overline{T}, -\overline{S}, -\overline{T}$ form the corners of a parallelogram, and, corresponding to the the value of ν which is the intersection of the line through $\overline{T}$ and $-\overline{S}$ with the horizontal axis, the line through $\overline{S}$ and $-\overline{T}$ meets the same axis in $-\nu$, hence in $-p \cdot \varepsilon / r$, (if both lines G, P are vertical) , or in $-b_P \cdot \varepsilon / b_G$ (if both lines G, P are not vertical) .

Corollary 1. If we reverse the roles of the $\mathbf{x}$-axis and $\mathbf{y}$-axis, we are able to formulate a corresponding statement: Consider the two parallel lines G, P , with the property that G does not meet $(0|0)$. Assume a fixed $\varepsilon \in \mathbb{R}$. Let us choose a point $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ on G , $\widehat{\mathbf{x}} \neq 0$, such that neither the line that connects $(0|0)$ and $S_v := (\widehat{\mathbf{x}}|\widehat{\mathbf{y}} - \varepsilon)$, nor the line that connects $(0|0)$ and $T_v := (\widehat{\mathbf{x}}|\widehat{\mathbf{y}} + \varepsilon)$ is parallel to G and P . We call $\overline{S_v}$ the projection of S_v on the line P, and $\overline{T_v}$ the projection of T_v on the line P. (That means that the three points $(0|0), S_v, \overline{S_v}$, and the three points $(0|0), T_v, \overline{T_v}$, respectively, are collinear, $\overline{S_v}, \overline{T_v} \in P$.) The four points $\overline{S_v}, \overline{T_v}, -\overline{S_v}, -\overline{T_v}$ are the corners of a parallelogram. We call $'\mu'$ the intersection of the line that connects $\overline{T_v}$ and $-\overline{S_v}$ with the vertical $\mathbf{y}$-axis , and we claim that the value of μ does not depend on the choice of $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ on G .

Proof. Trivial with the previous proposition. $\square$

Again we write down the last proposition in a seemingly more general form.

Proposition 4. Let us again take the euclidean space $\mathbb{R}^2$. Consider the two parallel lines G, P (P means 'projection line') , and an arbitrary third line ($\neq G$) that we call $Axis$. We fix a point $Origin$ on $Axis \backslash G$, and an $\varepsilon \geq 0$. Let us choose a point $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ on $G \backslash Axis$, and draw the straight line $\widehat{Axis}$, meeting $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ and parallel to $Axis$. We mark two unique points S, T on $\widehat{Axis}$, such that the distance of both to $(\widehat{\mathbf{x}}|\widehat{\mathbf{y}})$ is ε . We assume the extra property that the line that connects $Origin$ and S and the line that connects $Origin$ and

T are not parallel to G and P, respectively. Thus we are able to 'project' S and T onto P. We call $\overline{S}$ the projection of S on the line P, and $\overline{T}$ the projection of T on the line P, both projections relatively to $Origin$. (That means that the three points $Origin, S, \overline{S}$, and the three points $Origin, T, \overline{T}$, respectively, are collinear, $\overline{S}, \overline{T} \in P$.) Further we denote two points $-\overline{S}, -\overline{T}$, such that the four points $Origin, S, \overline{S}, -\overline{S}$, and the four points $Origin, T, \overline{T}, -\overline{T}$, respectively, are collinear, and the distance of $Origin$ and $\overline{S}$ is equal to the distance of $Origin$ and $-\overline{S}$, and the distance of $Origin$ and $\overline{T}$ is equal to the distance of $Origin$ and $-\overline{T}$, respectively. The four points $\overline{S}, \overline{T}, -\overline{S}, -\overline{T}$ form a parallelogram with centre $Origin$. We call $'\nu'$ the intersection of the line that connects $\overline{T}$ and $-\overline{S}$ with $Axis$, and we claim that ν does not depend on the choice of $(\hat{x}|\hat{y})$ on G. (See Picture 4).
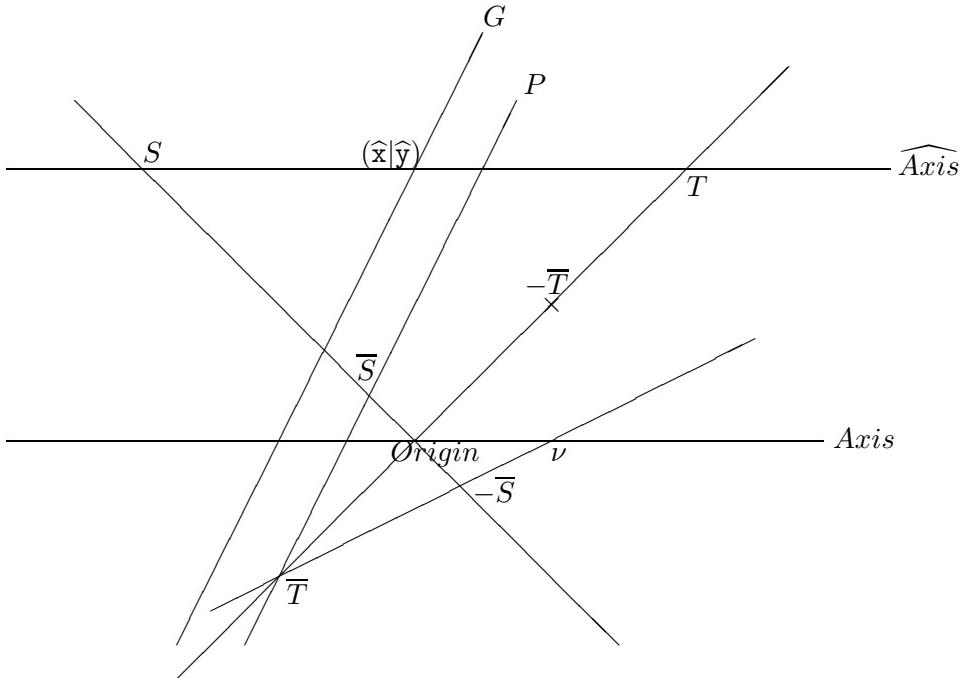
Picture 4

Lemma 3. We have that proposition 3 $\implies$ proposition 4.

Proof. With an easy transformation of coordinates, we get $(0|0) = Origin$, and x-axis = $Axis$. $\square$

Remark 3. For all propositions it would be desirable to have a construction with compass and ruler, using the classical methods of the 'old Greeks'.

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